FINITE PRESENTABILITY AND COHOMOLOGICAL PROPERTIES OF FIBRE PRODUCTS

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Abstract. We give conditions that characterize the existence of certain finitely presented normal fibre products in the direct product of a group by itself. Our results apply to normal fibre products of one-relator groups and right angled Artin groups. We show that higher cohomological finiteness properties (FP_n, FP_∞) are inherited by the the untwisted normal fibre product in a direct product of n copies of certain metabelian or virtually solvable groups.

1. Introduction

Even for classes of groups for which we have a good understanding of their lattices of finitely generated subgroups, such as for example finitely generated free groups, things change dramatically if one wants to look at subgroups of direct products. A common reduction when studying subgroups of direct products is to consider only subdirect products, i.e., subgroups that project epimorphically onto each factor since this kind of subgroups is all what one needs to know if one assumes knowledge of the lattices of subgroups of the factors. But even the full family of subdirect products is too wild in general. In a series of papers, started by Baumslag-Roseblade and continued by Baumslag, Bridson, Miller, Short and also Kochloukova several features of subdirect products of groups such as free, surface or limit groups are considered, with a special emphasis on the problem of determining which subdirect products are finitely presented or have certain cohomological finiteness properties. For example, in [3], [12], [13], [22] necessary conditions are given for a subdirect product to be of one of those types. And in [14] there is a sufficient condition for a subdirect product of arbitrary groups to be finitely presented (see also [22]). For a pro-p version of some of those results, see [23].

The 1-2-3 Theorem and its asymmetric version ([4], [14]) give also sufficient conditions for subdirect products to be finitely presented, but in this case restricted to the special case of fibre products. In this paper we are going to consider normal fibre products of a given group G by copies of itself. Let N be a normal subgroup of G and $p : G \to G/N$ the projection map.

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The untwisted N-fibre product in $G \times G$ is the subdirect product

$$H := \{(g_1, g_2) \in G \times G \mid p(g_1) = p(g_2)\}.$$

By [15] Proposition 1.2 H is normal in $G \times G$ if and only if $G' \leq N$, the same happens for products with more factors. Naively one could think that given its resemblance with the diagonal group, the untwisted fibre product should keep the properties of the ambient group G, and this is sometimes the case. For example, for $N = G'$, if G is metabelian and finite presented, then so is H ([5]). Given an automorphism ϕ of G/N, we can "twist" one of the components of H via ϕ and get a new group H_ϕ (see Section 2). It is also proven in [5] that in the metabelian case the group H_ϕ is not necessarily finitely presented although in some sense "almost always is" ([20]). But the situation is very different for other classes of groups: the results mentioned above imply that if G is free or a limit group then H is not finitely presented, nor can be any of the twisted H_ϕ. Here we show that other intermediate situations may happen. For example, there are finitely presented groups G for which the untwisted H for $N = G'$ is not finitely presented, however there is some ϕ automorphism of G/N such that H_ϕ is.

To do that we use Sigma theory to understand this phenomena. In the case when G is finitely presented and G/N has torsion free rank 1, we show that there is some finitely presented normal fibre product in $G \times G$ if and only if $\Sigma^1(G) \cap S(G, N) \neq \varnothing$ (Proposition 3.11, see Section 2 for notation) which is equivalent to the fact that the untwisted fibre product is finitely presented. However the situation is completely different if the torsion free rank of G/N is 2. We show (this is Theorem 3.12 below)

Theorem A: *Let N be a normal subgroup of the finitely presented group G with G/N of torsion free rank 2. Then there is some finitely presented N-fibre product in $G \times G$ if and only if there is some $[\chi] \in \Sigma^1(G) \cap S(G, N)$ with $[-\chi] \in \Sigma^1(G) \cap S(G, N)$. This happens if and only if $G =< t > \ltimes K$ where K is a finitely generated normal group with $N \leq K$.*

This result allows us to explore some classes of groups for which there is a nice way to compute $\Sigma^1(G)$ available. This is for example the case of non cyclic 1-relator groups. For these groups, either $\Sigma^1(G) = \varnothing$ ([27]) or they are 2-generated thus G/G' has torsion free rank at most 2 and $\Sigma^1(G)$ can be computed using what we call Brown's method. Using this fact and Theorem A is easy to provide examples of groups G for which the untwisted G'-fibre product in $G \times G$ is not finitely presented but there is some twisted one which is.

The situation is similar for right-angled Artin groups. There is also a nice way to compute $\Sigma^1(G_\Delta)$ in this case, where Δ is a flag complex and G_Δ the associated right angled Artin group ([27]) and we show.

Theorem B: *Let $G := G_\Delta$ be a right-angled-Artin group. Let $V(\Delta)$ be the set of vertices of Δ. Then there is some normal finitely presented G'-fibre product in $G \times G$ if and only if $|V(\Delta)| \leq 2|S|$ for any $S \subseteq V(\Delta)$ such the subcomplex of Δ obtained by removing the vertices in S is disconnected and S is minimal with respect to that property.*

To investigate what happens if one is interested in higher cohomological finiteness properties, we concentrate in the classes of metabelian and virtually solvable groups. For metabelian groups things are as expected, i.e., any normal untwisted fibre product has the same properties as the original group as long as one assumes that the group satisfies the FP_n-conjecture. This is a well known conjecture in metabelian groups, which has been proven in several cases (see Section 4). For the more general case of virtually solvable groups we consider only the property of being FP_∞. We prove (Theorem 4.2 below):

Theorem C: *Let G be virtually solvable of type FP_∞. Then for any $m > 0$ any untwisted normal fibre product H of $G \times \overset{m}{\dots} \times G$ is also of type FP_∞.*

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2. Preliminaries in fibre products, cohomological finiteness and Sigma theory

Let G be a group and put $G^n := G \times \ldots \times G$. A subdirect product in G^n is a subgroup $H \leq G^n$ such that the restriction to H of the projection onto each factor is an epimorphism. By [15] Proposition 1.2, a subdirect product H is normal in G^n if and only if $(G^n)' = (G')^n \leq H$.

Recall that a group G is said to be of type FP_m for $m \leq 0$ or $m = \infty$ if there is a resolution of the trivial module by projective modules which are finitely generated up to the m-th one. And it is of type $\mathrm{F_m}$ if it admits a model for the Eilenberg-Maclane space $K(G,1)$ with finite m-skeleton. Being finitely generated is equivalent to being F_1 or FP_1 and being finitely presented is equivalent to being of type F_2 so the properties $\mathrm{F_m}$ are usually considered as homotopical higher dimensional analogues of finite presentability.

Question 2.1. If G has cohomological type FP_m, for which normal subdirect products of G^n is the same true?

We are going to consider only a special kind of subdirect products

Notation 2.2. Let N be a normal subgroup of the group G and let p be the projection map $p : G \to G/N$. The untwisted N-fibre product in G^n is

$$H = \{(g_1, \ldots, g_n) \mid g_i \in G_i, p(g_1) = \ldots = p(g_n)\}.$$

Let $\phi_1, \ldots, \phi_n$ be automorphisms of $Q = G/N$. They yield $\phi = (\phi_1, \ldots, \phi_n)$ which is an automorphism of $G^n/(G^n)'$. The ϕ-twisted N-fibre product in G^n is

$$H_\phi = \{(g_1, \ldots, g_n) \mid g_i \in G_i, \phi_1(p(g_1)) = \ldots = \phi_n(p(g_n))\}.$$

Observe that if we put $\bar\phi = (1_Q, \phi_1^{-1}\phi_2, \ldots, \phi_1^{-1}\phi_n)$, then $H_\phi = H_{\bar\phi}$ so when considering twisted fibre products we will always assume that the first component of ϕ is trivial.

In the particular case when $\phi_1 = 1$ and ϕ_i, $i = 2\dots,n$ can all be lifted to automorphisms of G we have $H_\phi = \phi^{-1}(H) \cong H$.

By normal fibre product in G^n we refer to one of the groups above (twisted or untwisted) in the case when $G' \le N$.

Question 2.3. Assume that G is of type FP_m ($\mathrm{F_m}$) for some $0 < m$ or $m = \infty$. Which untwisted normal fibre products in G^n are of type FP_m ($\mathrm{F_m}$)? If for a given N the untwisted N-fibre product is not of type FP_m ($\mathrm{F_m}$), when is it true for some twisted N-fibre product in G^n?

Remark 2.4. Let G be finitely generated and $G' \le N$. The group $Q = G/N$ is finitely generated abelian. Put $Q = t(Q) \times L$ with $t(Q)$ the torsion subgroup and L torsion free. Let $\varphi \in \mathrm{Aut}Q$ and denote by $\bar\varphi$ its restriction to L, which we see as an automorphism of Q acting trivially on $t(Q)$. Let φ and $\bar\varphi$ be automorphisms of G and $\phi := (1,\varphi)$, $\bar\phi := (1,\bar\varphi)$. Then the subgroups H_ϕ and $H_{\bar\phi}$ of G^2 have the same intersection with the finite index subgroup $L \times L$ thus they are commensurable. As all the properties considered in Question 2.3, i.e., finite generation, finite presentability, types FP_m, $\mathrm{F_m}$ are preserved under finite index extensions, it suffices to consider normal fibre products associated to automorphisms of Q acting trivially on $t(Q)$.

It is easy to show that if some normal fibre product in G^n happens to be of type FP_m ($\mathrm{F_m}$), then the group G has the same property:

Lemma 2.5. *Let G be finitely generated and assume that for some $n, m > 0$ some normal fibre product H in G^n is of type FP_m ($\mathrm{F_m}$). Then G is also of type FP_m ($\mathrm{F_m}$).*

Proof. Observe that H is normal in G^n and G^n/H is a finitely generated abelian group, thus it is of type F_∞. The result for FP_m follows with the same proof as in [6] Prop. 2.7. For $\mathrm{F_m}$, see [18], Section 7.2 Ex. 1. □

The main tool that we are going to use is Sigma theory so we recall here the main definitions needed. A character of a group G is a homomorphism $\chi : G \to \mathbb{R}$ where $\mathbb{R}$ is seen as an additive group. Given a character χ we put $[\chi] = \{t\chi \mid 0 < t \in \mathbb{R}\}$ and let

$$S(G) := \{[\chi] \mid 0 \ne \chi : G \to \mathbb{R}\}$$

which is often useful to visualize as an $k-1$-sphere where k is the torsion free rank of G/G'. If $N \le G$ is a subgroup such that $G' \le N$, we set $S(G,N) := \{[\chi] \in S(G) \mid \chi|_N = 0\}$. Given $[\chi] \in S(G)$, consider the monoid $G_\chi := \{g \in G \mid \chi(g) \ge 0\}$. Let $m \ge 0$ or $m = \infty$. If G is of type FP_m, the homological m-th Bieri-Neumann-Renz-Strebel invariant (or m-th Sigma invariant for short), first defined in [9] is

$$\Sigma^m(G,\mathbb{Z}) := \{[\chi] : G_\chi \text{ is of type } \mathrm{FP}_m\}$$

for the obvious generalization for monoids of the condition of being FP_∞.

If G is FP_∞ we have

$$S^{n-1} \supseteq \Sigma^0(G,\mathbb{Z}) \supseteq \dots \supseteq \Sigma^m(G,\mathbb{Z}) \supseteq \dots$$

and $\Sigma^\infty(G,\mathbb{Z}) = \cap_m \Sigma^m(G,\mathbb{Z})$.

If G is of type $\mathrm{F_m}$, then one can define the homotopical analog $\Sigma^m(G)$ (first defined in [26]). Some of the most remarkable features of these invariants are that they are open subsets of $S(G)$ ([9]) and that they provide information about which subgroups over G' are also of type FP_m (resp. $\mathrm{F_m}$).

Theorem 2.6. *([9]) Let G be of type FP_m (resp. $\mathrm{F_m}$) and $G' \leq N \leq G$. Then N is of type FP_m ($\mathrm{F_m}$) if and only if*

$$S(G,N) \subseteq \Sigma^m(G,\mathbb{Z}).$$

(This result will be heavily used throughout the paper.)

Definition 2.7. Let $\Omega, \Gamma \subseteq S(G)$ be subsets of the character sphere. We say that Ω is 2-tame if for any $[\chi], [\nu] \in \Omega$, $\chi + \nu \neq 0$ (in particular, the empty set is 2-tame). And we say that Ω and Γ are relatively tame if for any $[\chi] \in \Omega$, $[\mu] \in \Gamma$ we have $\chi + \mu \neq 0$.

Given two groups T and G, we may identify $S(G \times T)$ with $S(G) * S(T)$ where, for $A \subseteq S(G)$ and $B \subseteq S(T)$,

$$A * B := (A \times B) \cup A \cup B.$$

Assume that both T and G are of type FP_m or $\mathrm{F_m}$. The following formulas are known as Meiner's inequalities ([7] Theorem 1.2)

$$\Sigma^m(G \times T, \mathbb{Z})^c \subseteq \bigcup_{i=0}^{m} \Sigma^i(G,\mathbb{Z})^c * \Sigma^{m-i}(T,\mathbb{Z})^c,$$

$$\Sigma^m(G \times T)^c \subseteq \bigcup_{i=0}^{m} \Sigma^i(G)^c * \Sigma^{m-i}(T)^c.$$

The homotopical and homological invariants are connected via

$$\Sigma^m(G \times T) = \Big(\Sigma^m(G \times T, \mathbb{Z}) \smallsetminus (S(G) \cup S(T))\Big) \cup \Sigma^m(G) \cup \Sigma^m(T)$$

(this is [7] Theorem 1.1).

By [7] Theorem 1.5 (first proven by Schütz), Meiner's inequalities are equalities if $m \leq 3$. Since we will be mostly interested in $m = 2$, we will make more explicit the above formulas in this case. Thus assuming that G, T are of type FP_2,

$$\Sigma^2((G \times T, \mathbb{Z})^c = (\Sigma^1(G)^c \times \Sigma^1(G)^c) \cup (\{0\} \times \Sigma^2(T,\mathbb{Z})^c) \cup (\Sigma^2(G,\mathbb{Z})^c \times \{0\}).$$

We set

$$\Omega := \{(\chi, \mu) \mid \chi, \mu \neq 0, \chi \in \Sigma^1(G) \text{ or } \mu \in \Sigma^1(T)\},$$

then we have

$$\Sigma^2(G \times T, \mathbb{Z}) = \Omega \cup (\{0\} \times \Sigma^2(T,\mathbb{Z})) \cup (\{0\} \times \Sigma^2(G,\mathbb{Z})). \tag{1}$$

and if G, T are finitely presented,

$$\Sigma^2(G \times T) = \Omega \cup (\{0\} \times \Sigma^2(T)) \cup (\{0\} \times \Sigma^2(G)). \tag{2}$$

Although we have tried to provide all the relevant references for the results used, for anything related to the invariants $\Sigma^1(G,\mathbb{Z})$, $\Sigma^1(G)$ the reader is referred to the excellent survey [27].

3. Finitely generation and finite presentability of normal fibre products

As one can expect, the smaller the corank of a fibre product is, the closer properties to that of the ambient group one can get. This is formalized in the next result.

Lemma 3.1. *Let N be a normal subgroup of G with $G' \leq N$ and let φ be an automorphism of G/N acting trivially on $t(G/N)$. Then there is a lift of φ to an automorphism $\hat{\varphi}$ of G/G' so that if $\varphi = 1$, then $\hat{\varphi} = 1$ and a short exact sequence*

$$1 \to H_{(1,\hat{\varphi})} \to H_{(1,\varphi)} \to Q_1 \to 1$$

with Q_1 finitely generated abelian. Therefore if $H_{(1,\hat{\varphi})}$ is of type FP_m *or* $\mathrm{F_m}$ *for some $0 \leq m$ or $m = \infty$, then so is $H_{(1,\varphi)}$.*

Proof. Put $T/N := t(G/N)$. As G/T is free abelian and G/G' maps onto it, there is some L/G' with $G/G' = L/G' \times T/G'$. As φ is an automorphism of G/T, we can use this decomposition to extend it to an automorphism $\hat{\varphi}$ of G/G' acting trivially on T/G' (and in particular acting trivially in the torsion part of G/G'). Then $H_{(1,\hat{\varphi})} \subseteq H_{(1,\varphi)}$. As $(G^n)' \leq H_{(1,\hat{\varphi})}$, we have a group extension

$$1 \to H_{(1,\hat{\varphi})} \to H_{(1,\varphi)} \to Q_1 \to 1$$

with Q_1 an abelian finitely generated group. Therefore the assertion follows by [6] Prop. 2.7, [18] Theorem 7.2.21, Ex. 1 pg. 176. □

In particular, if the untwisted G'-fibre product is of type FP_m or $\mathrm{F_m}$, then so is any untwisted normal fibre product.

For $m = 1$, i.e., for finite generation the answer to Question 2.3 is easy:

Lemma 3.2. *Let G be finitely generated and $n > 0$. Then any normal fibre product in G^n is finitely generated.*

Proof. By Lemma 3.1 we may assume that $N = G'$. Consider first the case when $n = 2$ and H is the untwisted G'-fibre product in G^2. Fix a finite generating system $\{x_1, \ldots, x_s\}$ of G. For $i = 1, \ldots, s$ let $g_i := (x_i, 1), h_i := (1, x_i), t_i := (x_i, x_i) \in G^2$. We claim that the finite family

$$\{t_i, [g_i^{\pm 1}, g_j^{\pm 1}], [h_i^{\pm 1}, h_j^{\pm 1}]\}_{1 \leq i,j \leq s}$$

generates H. Observe first that all of its elements lie in H. Moreover, since the elements $t_i(G^2)'$ generate $H/(G^2)'$, the claim will follow if we show that $(G^2)' = (G')^2$ is H-generated by $\{[g_i, g_j], [h_i, h_j]\}_{1 \leq i,j \leq s}$ (i.e., if we show that the smallest subgroup of $(G')^2$ which is normal in H and contains that family is the whole $(G^2)')$. But this is a consequence of the well known fact that G' is G-generated by $\{[x_i, x_j]\}_{1 \leq i,j \leq s}$. (Which in turn follows from the equations $[gx_i, h] = x_i^{-1}[g, h]x_i[x_i, h]$).

Now, let φ be an automorphism of $Q = G/G'$ acting trivially on $t(Q)$. Denote by r the torsion free rank of Q, then $r \leq s$ and we may assume that $\{x_1T, \ldots, x_rT\}$ is a generating system for $G/T = Q_1 := Q/t(Q)$. If we write Q_1 additively, φ can be expressed using an invertible integer matrix $A = (a_{ij})$. With multiplicative notation we have $\varphi(x_iT) = x_1^{a_{i,1}} \ldots x_r^{a_{i,r}}T$. Let $y_i := x_1^{a_{i,1}} \ldots x_r^{a_{i,r}}$ for $1 \leq i \leq r$ and if $r < s$ set $y_i = x_i$ for $r < i \leq s$.

We claim that $\{y_1, \ldots, y_s\}$ is also a generating system of G. Since A can be written as a product of integer elementary matrices, by writing φ as composition of the associated maps, we see that it suffices to consider the case when there are $1 \le u \ne v \le r$ and $t \in \mathbb{Z}$ such that $y_i = x_i$ for $i \ne v$ and $y_v = x_v x_u^t$. But then $x_v = y_v x_u^{-t} = y_v y_u^{-t}$ so the claim follows. Now, the obvious modification of the first part of the proof (taking $t_i = (y_i, x_i)$, $g_i = (y_i, 1)$ and $h_i = (1, x_i)$) yields a generating family of $H_{(1,\varphi)}$. The case of an arbitrary n follows exactly in the same way. □

Remark 3.3. In general, applying naively an automorphism of an abelian non-torsion free quotient to a family of generators does not yield a new family of generators. A trivial example is $C_\infty =< t >$, $N =< t^3 >$. In $G/N = C_3$ we have the automorphism $tN \mapsto t^2N$ but t^2 does not generate the whole group.

Let φ be an automorphism of $Q = G/N$ acting trivially on $t(Q)$. Then φ acts on $S(Q) = S(Q/t(Q))$ via $\varphi(\chi) := \chi \circ \varphi^{-1}$. As $S(Q)$ and $S(G,N)$ can be naturally identified via the projection map $p : G \to Q$, this yields an action of φ on $S(G,N)$ (another way to argue is to proceed as in the proof of Lemma 3.1 and consider the action of $\hat{\varphi}$ on the whole $S(G)$, which preserves $S(G,N)$ setwise). To avoid unnecessary complications, we are not going to distinguish in what follows between $S(G,N)$, i.e., characters of G vanishing in N and $S(Q)$, i.e., characters of Q. In the particular case when φ can be lifted to an automorphism of G, it preserves the subset $\Sigma^1(G) \cap S(G,N)$, but this is not true in general.

Theorem 3.4. *Let N be a normal subgroup of the finitely presented (resp.* FP_2*) group G, φ an automorphism of $Q := G/N$ acting trivially on $t(Q)$ and $\phi := (1, \varphi)$. Then the ϕ-twisted N-fibre product of G^2, H_ϕ, is finitely presented (resp.* FP_2*) if and only if $\Sigma^1(G)^c \cap S(G,N)$ and $\varphi(\Sigma^1(G)^c \cap S(G,N))$ are relatively tame.*

Proof. As $(G^2)' \le H_\phi$, [9] Remark 6.5 and Theorem B imply that H_ϕ is finitely presented if and only if $S(G^2, H_\phi) \subseteq \Sigma^2(G^2)$ and the analogous statement in the FP_2 case. A first obvious observation is that $S(G^2, H_\phi) \subseteq S(G,N) * S(G,N)$. Let $[\chi] = [(\chi_1, \chi_2)] \in S(G,N) * S(G,N)$ and let $g_1 \in G$, $q := g_1 N \in Q$. Choose $g_2 \in p^{-1}(\phi^{-1}(q))$, then $(g_1, g_2) \in H_\phi$ thus

$$\chi(g_1, g_2) = \chi_1(g_1) + \chi_2(g_2) = \chi_1(g_1) + \chi_2(\varphi^{-1}(q)) = (\chi_1 + \varphi(\chi_2))(g_1).$$

Therefore $H_\phi \le \mathrm{Ker}\chi$ if and only if $\chi_1 + \varphi(\chi_2) = 0$. This means that H_ϕ is finitely presented if and only if for any pair $[(\chi_1, \chi_2)] \in S(G,N) * S(G,N)$ such that $\chi_1 + \varphi(\chi_2) = 0$, we have $[(\chi_1, \chi_2)] \in \Sigma^2(G^2)$. By (2), $[(\chi_1, \chi_2)] \in \Sigma^2(G^2)$ if either $[\chi_1] \in \Sigma^1(G)$ or $[\chi_2] \in \Sigma^1(G)$. Thus H_ϕ is finitely presented if and only if for any pair $[(\chi_1, \chi_2)] \in S(G,N) * S(G,N)$ such that $\chi_1 + \chi_2 = 0$, we have either $[\chi_1] \in \Sigma^1(G)$ or $[\chi_2] \in \varphi(\Sigma_1(G))$. □

Corollary 3.5. *Let N be a normal subgroup of the finitely presented group G (resp.* FP_2*). Then the untwisted N-fibre product of G^2 is finitely presented (resp.* FP_2*) if and only if $\Sigma^1(G)^c \cap S(G,N)$ is 2-tame.*

Example 3.6. Let G be a finitely presented group with no non abelian free subgroups. Then for any quotient T of G, $\Sigma^1(T)^c$ is 2-tame. This is

Corollary B 1.10 in [27]. See also Theorems A 5.1 and A 5.15. So we deduce that for such groups T any untwisted normal fibre product in T^2 is finitely presented. This happens for example for any quotient of any subgroup of the Thompson group F.

Example 3.7. Let G be a group with G' finitely generated. Then $\Sigma^1(G) = S(G)$ thus $\Sigma^1(G)^c = \varnothing$ is 2-tame and in fact m-tame for any m. Then Theorem 3.4 implies that any normal fibre product of G^2 is finitely presented, moreover, in this case any normal subdirect product is. This is a particular case of the 1-2-3 Theorem since in this case the quotient group $Q = G/G'$ is of type F_∞.

In the previous two examples, the hypothesis of Corollary 3.5 are satisfied. Moreover, Example 3.7 is an extreme case. Now we consider the other extreme case.

Example 3.8. Let G be a free non abelian group. Then $\Sigma^1(G) = \varnothing$ ([27] A 2.1a 3)) thus for any N normal in G with $G' \leq N$ and G/N of rank at least one, $\Sigma^1(G)^c \cap S(G,N) = S(G,N)$ which is not 2-tame. The same happens for the following groups ([27] Proposition B3.2): Let Γ be a finitely generated group that admits a presentation with $k \geq 2$ generators and n relators. Let G be a quotient $G = \Gamma/M$ with $M \leq \Gamma''(\Gamma')^p$ for some prime p. Assume that

(i) either $n < k-1$,
(ii) or $n = k-1$, one relator is a proper power, say $r = w^k$, and p is a prime number dividing k.

Then $\Sigma^1(G) = \varnothing$. Therefore no normal fibre product of G^2 and more in general, no normal subdirect product can be finitely presented unless it has finite index in G^2. As a particular case, we see that for any 1-relator group G, $\Sigma^1(G) = \varnothing$ unless G is cyclic or 2-generated.

Remark 3.9. If G is a free or limit group, then $\Sigma^1(G) = \varnothing$ ([27], [22]) so we deduce that no normal fibre product in G^2 can be finitely generated. But by [12], [13], [22] a much stronger result holds since in this case no subdirect product can be finitely presented unless it has finite index in G^2.

Lemma 3.10. *Let $N < G$ be a normal subgroup of the finitely presented group G such that $\Sigma^1(G)^c \cap S(G,N)$ is not 2-tame. Then there is some finitely presented twisted N-fibre product in G^2 if and only is there is some φ automorphism of G/N such that $\varphi(\Sigma^1(G)^c \cap S(G,N)) \subseteq \Sigma^1(G) \cap S(G,N)$.*

Proof. Let $\phi = (1, -\varphi)$. We claim that the ϕ-twisted N-fibre product of G is finitely presented. To see it we only have to check that $\Sigma^1(G)^c \cap S(G,N)$ and $-\varphi(\Sigma^1(G)^c \cap S(G,N))$ are relatively tame. But since $-\varphi(\Sigma^1(G)^c \cap S(G,N)) \subseteq -\Sigma_1(G) \cap S(G,N)$, this is obvious (if $[\chi] \in \Sigma^1(G)^c \cap S(G,N)$ and $[\mu] \in \Sigma_1(G) \cap S(G,N)$ are such that $\chi - \mu = 0$, we would have $\chi = \mu$ which is a contradiction).

Conversely, if the ϕ-twisted N-fibre product of G is finitely presented, then $\Sigma^1(G)^c \cap S(G,N)$ and $-\varphi(\Sigma^1(G)^c) \cap S(G,N)$ are relatively tame. If there is some $[\chi] \in \varphi(\Sigma^1(G)) \cap \Sigma^1(G)^c \cap S(G,N)$, then $[-\chi] \in -\varphi(\Sigma^1(G)^c) \cap S(G,N)$ and we would have a contradiction. Thus $\varphi(\Sigma^1(G) \cap S(G,N)) \subseteq \Sigma^1(G) \cap S(G,N)$. $\square$

Let N be a normal subgroup of G with $G' \leq N$. Let k be the torsion free rank of $Q := G/N$. If φ is an automorphism of Q acting trivially on $t(Q)$, it obviously yields a linear map of $\mathbb{R}^k$ which preserves the lattice $\mathbb{Z}^k$. Conversely, any linear map of $\mathbb{R}^k$ inducing a bijection in $\mathbb{Z}^k$ can be seen as coming from an automorphism of $Q/t(Q)$. Fix $q_1, \ldots, q_k$ generating $Q/t(Q)$, we can identify $\mathbb{R}^k$ with the set of characters $\chi : G \to \mathbb{R}$ with $\chi|_N = 0$. Under this identification the vector $v = (v_1, \ldots, v_k)$ corresponds to the character χ with $\chi(q_i) = v_i$. Then for $\chi \neq 0$ the class $[\chi]$ is identified with the ray from the origin containing the vector v which we denote $[v] = \{\lambda v \mid \lambda > 0\}$ and

$$S(Q) = \{[v] \mid 0 \neq v \in \mathbb{R}^k\}.$$

For any $\Omega \subseteq \mathbb{R}^k$, let

$$[\Omega] = \{[v] \mid 0 \neq v \in \Omega\}.$$

3.1. **N-fibre products with G/N of small rank.**

Proposition 3.11. *Let N be a normal subgroup of the finitely generated group G with G/N of torsion free rank 1. The following conditions are equivalent.*

i) *The untwisted N-fibre product in G^2 is finitely presented.*
ii) *There is some finitely presented N-fibre product in G^2.*
iii) $\Sigma^1(G) \cap S(G, N) \neq \varnothing$.

Proof. Obviously, i) implies ii). If ii) holds, then $\Sigma^1(G) \cap S(G, N) \neq \varnothing$ by, for example, Theorem 3.4. Finally, if $\Sigma^1(G) \cap S(G, N) \neq \varnothing$, then, since G/N has torsion free rank 1, $S(G, N)$ has only two points so $\Sigma^1(G)^c \cap S(G, N)$ is 2-tame. We get i) by Corollary 3.5. □

Theorem 3.12. *Let N be a normal subgroup of the finitely generated group G with G/N of torsion free rank 2. The following conditions are equivalent.*

i) *There is some finitely presented N-fibre product in G^2.*
ii) *There is some $[\chi] \in \Sigma^1(G) \cap S(G, N)$ with $[-\chi] \in \Sigma^1(G) \cap S(G, N)$.*
iii) *There is some $[\chi] \in \Sigma^1(G) \cap S(G, N)$ discrete with $[-\chi] \in \Sigma^1(G) \cap S(G, N)$.*
iv) $G = < t > \ltimes K$ *where K is a finitely generated group with $N \leq K$.*

Proof. The equivalence between iii) and iv) follows using [27] Corollary A4.3 (take $K = \mathrm{Ker}\chi$, t with $\chi(t) = 1$, observe that since $[\chi] \in S(G, N)$ this implies $N \leq K$), Proposition B2.9, Theorem B3.5 and Proposition B3.15. Obviously, iii) implies ii) and the converse follows from the fact that the rational numbers are dense in $\mathbb{R}$, thus the discrete characters are also dense in $\Sigma^1(G) \cap S(G, N)$ (recall that $\Sigma^1(G)$ is open, see [27] Theorem A3.3).

Assume now i). In the case when $\Sigma^1(G)^c \cap S(G, N)$ is not 2-tame, there are $[\chi], [-\chi] \in \Sigma^1(G)^c \cap S(G, N)$. By Lemma 3.10 there is some φ automorphism of $\mathbb{Z}^2$ such that $\varphi(\Sigma^1(G)^c \cap S(G, N)) \subseteq \Sigma^1(G) \cap S(G, N)$. Then for $\mu := \varphi(\chi)$, $[\mu], [-\mu] \in \Sigma^1(G) \cap S(G, N)$ so we have ii). So we are left with the case when $\Sigma^1(G)^c \cap S(G, N)$ is 2-tame. Let $[\chi] \in S(G, N)$ be in the boundary of the closed set $\Sigma^1(G)^c \cap S(G, N)$. By tameness, $[-\chi] \in \Sigma^1(G) \cap S(G, N)$. Taking into account again that $\Sigma^1(G)$ is open we see that we may choose a $[\mu] \in \Sigma^1(G) \cap S(G, N)$ "close" to $[-\chi]$ so that $[-\mu] \in \Sigma^1(G) \cap S(G, N)$.

Finally, we assume that iii) holds and want to prove i). We may assume that $\Sigma^1(G)^c \cap S(G,N)$ is not 2-tame. Choose a vector $v = (v_1, v_2) \in \mathbb{Z}^2$ such that $[v]$ corresponds to $[\chi]$ (the components v_1 and v_2 lie in $\mathbb{Z}$ because χ is discrete). By applying a suitable automorphism of $\mathbb{Z}^2$ if necessary, we may assume that $0 < v_1, v_2$. We may also choose v so that v_1, v_2 are coprime so there exist integers s, t such that $v_1 s - v_2 t = 1$. Let m be an integer and put $w_1 = t + mv_1$, $w_2 = s + mv_2$, then $v_1 w_2 - v_2 w_1 = 1$. Therefore

$$|\frac{w_1}{w_2} - \frac{v_1}{v_2}| = |\frac{1}{v_2(s + mv_2)}|$$

which means that if $[\mu]$ is the character associated to $w := (w_1, w_2)$, then making m big we can make $[\chi], [\mu]$ as close as we want. In particular, since $\Sigma^1(G)$ is open, we can choose an m so that the whole closed arcs between $[\chi]$ and $[\mu]$ and between $[-\chi]$ and $[-\mu]$ lie in $\Sigma^1(G) \cap S(G,N)$. Let φ be the linear map defined by $\varphi(v) = -w$, $\varphi(w) = v$. Then from the fact that the matrix with columns w, v is an invertible integer matrix follows easily that φ restricts to a bijection of $\mathbb{Z}^2$. Consider the following open subsets of $\mathbb{R}^2$:

Ω_1 bounded by the rays from the origin containing w and $-v$,

Ω_2 bounded by the rays from the origin containing $-w$ and v.

Then $\Sigma^1(G)^c \cap S(G,N) \subseteq [\Omega_1 \cup \Omega_2] \cap S(G,N)$ and $\varphi([\Omega_1 \cup \Omega_2] \cap S(G,N)) \subseteq \Sigma^1(G) \cap S(G,N)$. Using Lemma 3.10 we have i).

□

3.2. **One relator groups.** We are going to apply Theorem 3.12 to one-relator groups. Note that by Remark 3.8, we only have to consider the case of 2-generated 1-relator groups, since if the number of generators is bigger, then $\Sigma^1(G)$ is empty and if the group is cyclic, $\Sigma^1(G) = S(G)$. For these groups we have a very nice way to compute $\Sigma^1(G)$ known as Brown's algorithm that can be found in [27].

Our first result deals with the case when the relator has the form $u = v$ where u, v are positive words on the generators so that the sum of the exponents of each of the generators in u equals that in v. These groups have been considered by Baumslag in [2] (without the assumption of being 2-generated), see also [27] B4.2a. Note that the condition on the exponents implies that G/G' has rank 2.

Theorem 3.13. *Let G be a 2-generated 1-relator group such that the relator has the form $u = v$ where u, v are positive words on the generator so that the sum of the exponents of each of the generators in u equals that in v. Then there is some finitely presented twisted G'-fibre product in G^2.*

Proof. Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ denote the first, second, third and fourth open quadrants of $\mathbb{R}^2$ i.e., $u_1, u_2 > 0$ for $(u_1, u_2) \in \Omega_1$, $u_1 < 0$, $u_2 > 0$ for Ω_2, $u_1, u_2 < 0$ for Ω_3 and $u_1 > 0$, $u_2 < 0$ for Ω_4. Using Brown's procedure (see [27] B4.2a) it is easy to prove that $[\Omega_1 \cup \Omega_3] \subseteq \Sigma^1(G)$. Let $(x_1, x_2) \in \Omega_1$ and consider the character χ with $\chi(a) = x_1$, $\chi(b) = x_2$. As $x_1, x_2 > 0$, if we denote by $u_1, u_2, \ldots, u$ resp. $v_1, v_2, \ldots, v$ the initial resp. final subwords of length $1, 2, \ldots$ of u resp. v, we have

$$\chi(u_1) < \chi(u_2) < \ldots < \chi(u) > \chi(uv_1^{-1}) > \chi(uv_2^{-1}) > \ldots \chi(uv^{-1}) = 0.$$

As χ reaches its minimum exactly once in this sequence (since $0 < \chi(u_1)$), we deduce $\chi \in \Sigma^1(G)$. Now, let χ be the character associated to $(x_1, x_2) \in \Omega_3$, i.e., $x_1, x_2 < 0$. When we apply χ to $u_1, u_2, \ldots, u, uv_1^{-1}, \ldots, uv^{-1}$ we obtain the same as before but reversing all the inequalities. Thus now $\chi(u)$ is the minimum and it is reached exactly once thus $\chi \in \Sigma^1(G)$. This means that the result follows from Theorem 3.12. But in this case it is easy to find a suitable φ more explicitly.

Let φ_1 be the automorphism of $\mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\varphi_1(\overline{\Omega_2}) \subseteq \overline{\Omega_3}$ ($\overline{\Omega_i}$ is the closure of Ω_i) and $\varphi_1(\overline{\Omega_4}) \subseteq \overline{\Omega_1}$. Let $w_1 := (2,1)$ and $w_2 := (1,1)$. Let φ_2 be the automorphism of $\mathbb{Z}^2$ given by $(1,0) \mapsto w_1$, $(0,1) \mapsto w_2$. Then φ_2 is an automorphism of $\mathbb{Z}^2$ with $\varphi_2(\overline{\Omega_1}) \subseteq \Omega_1$ and $\varphi_2(\overline{\Omega_3}) \subseteq \Omega_3$. Finally, put $\varphi := \varphi_2 \circ \varphi_1$. Then

$$\varphi(\Sigma^1(G)^c) \subseteq \varphi_2 \circ \varphi_1([\overline{\Omega_2} \cup \overline{\Omega_4}]) \subseteq \varphi_2([\overline{\Omega_1} \cup \overline{\Omega_3}]) \subseteq [\Omega_1 \cup \Omega_3] \subseteq \Sigma^1(G).$$

□

Example 3.14. Let

$$G = < a, b \mid aba^2b = ba^2ba > .$$

Using Brown's procedure it is easy to check that

$$\Sigma^1(G) = \{[\chi], [-\chi]\}$$

with $\chi(a) = -1$, $\chi(b) = 2$. Obviously, it is not 2-tame thus the untwisted G'-fibre product in G^2 is not finitely presented. However, for any φ automorphism of $\mathbb{Z}^2$ such that $\varphi(-1,2) \neq (-1,2), (1,-2)$, the corresponding twisted G'-fibre product is finitely presented.

Proposition 3.15. *Let G be a 2-generated 1-relator group with generators a, b and relator r and assume that G/G' has torsion free rank 2. Then there is a finitely presented G'-fibre product in G^2 if and only if there is some discrete character χ of G with $\chi(a), \chi(b) \neq 0$ such that if r_1, $r_2, \ldots, r$ are the initial subwords of r, the sequence $\chi(r_1), \chi(r_2), \ldots, \chi(r)$ reaches both its maximum and its minimum exactly once.*

Proof. Note that the maximum in the sequence $\chi(r_i)$ is precisely the minimum of the same sequence but with $-\chi$ instead. By Brown's procedure the existence of such χ is equivalent to the existence of a discrete character with $\chi(a), \chi(b) \neq 0$ and with $[\chi], [-\chi] \in \Sigma^1(G)$. Assume that there is some μ discrete with $[\mu], [-\mu] \in \Sigma^1(G)$, the fact that $\Sigma^1(G)$ is open implies that there is some χ in the same conditions and with $\chi(a), \chi(b) \neq 0$. So the result follows by Theorem 3.12. □

3.3. **Right angled Artin groups.** Let Δ be a flag complex and G_Δ the associated right angled Artin group. If $S \subseteq V(\Delta)$ is a subset of vertices we denote by Δ_S the smallest subcomplex of Δ containing S and set $G_S := G_{\Delta_S}$ seen as a subgroup of G. By [27] Proposition A4.14 if G is not abelian,

$$\Sigma^1(G)^c = \bigcup_{S \in \mathcal{S}} S(G, G_S) \tag{3}$$

where $\mathcal{S}$ is the set of subsets $S \subseteq V(\Delta)$ such that the subcomplex of Δ obtained by removing the vertices in S is disconnected and S is minimal with respect to that property.

Theorem 3.16. *Let $G := G_\Delta$ be a right-angled-Artin group. Then*

i) *The untwisted G'-fibre product of G^2 is finitely presented if and only if G is abelian.*

ii) *Let $k = |V(\Delta)|$. Then there is some twisted finitely presented G'-fibre product in G^2 if and only if for any $S \in \mathcal{S}$, $k \leq 2|S|$.*

Proof. For i) observe that for any $S \subseteq \mathcal{S}$, $S(G, G_S)$ is never 2-tame. Now, if G is abelian, the result is obvious. Conversely, assume $\mathcal{S} = \varnothing$. If there were two vertices $s_1, s_2 \in V(\Delta)$ which are not connected then the removal of all the other vertices would yield a disconnected complex which is a contradiction. Thus all the vertices are pairwise connected and therefore G is abelian.

For ii), note that Lemma 3.10 implies that there is a finitely presented twisted G'-fibre product in G^2 if and only if there is some φ automorphism of $\mathbb{Z}^k$ such that $\varphi(\Sigma^1(G)^c) \subseteq \Sigma^1(G)$. Put $\mathcal{S} = \{S_1, \ldots, S_t\}$ and denote by U_i the subvector space of $\mathbb{R}^k$ generated by $S(G, G_{S_i})$. Our condition is equivalent to the existence of φ automorphism of $\mathbb{Z}^k$ such that

$$\text{for any } 1 \leq i, j \leq t, \varphi(U_i) \cap U_j = 0. \tag{4}$$

As $k - |S_i| = \mathrm{dim} U_i = \mathrm{dim}\varphi(U_i)$, if (4) holds true, then $2k - |S_i| - |S_j| \leq k$ thus $k \leq |S_i| + |S_j|$. By choosing $i = j$ we get the condition in b).

So now we assume that $k \leq 2|S|$ for any $S \in \mathcal{S}$, i.e., that $2\mathrm{dim}U_i \leq k$ for $i = 1, \ldots, t$. We claim that there is some φ automorphism of $\mathbb{Z}^k$ as in (4). If we denote $T_i = V(\Delta) \smallsetminus S_i$, then U_i is the vector space generated by $S(G_{T_i})$. Note also that we may label the canonical basis of $\mathbb{R}^k$ with the elements in $V(\Delta)$ and under this labeling, U_i is generated precisely by those vectors corresponding to the vertices in T_i. Enlarging each U_i if necessary, we may assume that for $i = 1, \ldots, t$, $r := \mathrm{dim}U_i = \lfloor k/2 \rfloor$ where $\lfloor k/2 \rfloor$ is the integer part of $k/2$ and keep the property of being spanned by vectors in the canonical basis. We may moreover enlarge the family $\{U_1, \ldots, U_t\}$ to include all the r-dimensional subspaces generated by a subset of r vectors inside the canonical basis, since (4) for the enlarged set implies (4) for the original one. In particular, we assume now that t is the number of subsets of $\{1, \ldots, k\}$ consisting of precisely r elements.

By Lemma 3.17 below there is an integer $k \times k$-matrix A with $\mathrm{det}A = 1$ having all its minors positive. Let φ be the automorphism of $\mathbb{Z}^k$ with associated matrix A in the canonical basis. We claim that this φ satisfies condition (4). To do not have to distinguish between the cases when k is odd or even it will be convenient to denote by $\{V_1, \ldots, V_s\}$ the set $\{U_1, \ldots, U_t\}$ if k is even, and the set of all subspaces spanned by a subset of $r + 1$ vectors in the canonical basis if k is odd. Then in both cases if

$$V_i \cap \varphi(U_j) = 0 \text{ for any } 1 \leq i \leq s, 1 \leq j \leq t, \tag{5}$$

condition (4) will follow. For any $1 \leq i \leq s$, $1 \leq j \leq t$, let $A_{i,j}$ be the $k \times k$-matrix having as first columns the vectors generating V_i in their natural order and as last r columns the result of multiplying A by the vectors

generating U_j again in their natural order. Obviously, (5) is equivalent to $\det A_{i,j} \neq 0$ for any $1 \leq i \leq s$, $1 \leq j \leq t$. But one easily checks that each $\det A_{i,j}$ is a minor of the matrix A thus $\det A_{i,j} > 0$.

□

Lemma 3.17. *For any $k > 0$ there exists a symmetric integer $k \times k$-matrix A_k such that all its minors are positive and $\det A_k = 1$.*

Proof. We proceed by induction, the case $k = 1$ being obvious. Now, assume we have already A_k. We claim that there are integers $a_1, \dots, a_k, b$ such that the matrix

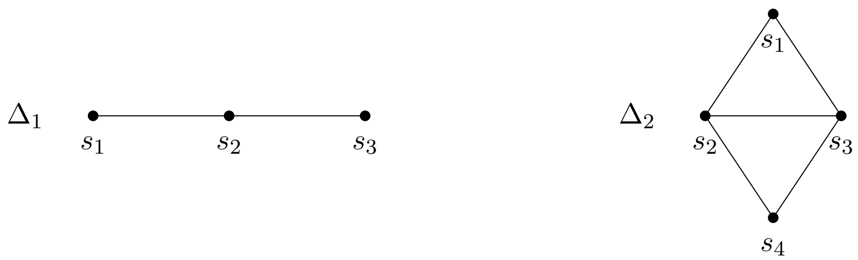
$$A_{k+1} = \left(\begin{array}{ccc|c} & & & a_1 \\ & A_k & & \cdot \\ & & & \cdot \\ & & & a_k \\ \hline a_1 & \cdot & \cdot \;\; a_k & b \end{array} \right)$$

has the desired properties. We need all the minors of A_{k+1} to be positive but by [17] Theorem 4.1, it suffices to consider the minors of a very special form, namely, those minors obtained from consecutive rows and columns which include either the first row or the first column (called row-initial or column-initial minors). By induction, all are positive except of possibly those in the upper right corner and those obtained by transposition of these ones. Obviously, we may choose $a_1 > 0$, then a_2 so that the 2×2-minor in the right upper corner is positive and so on until we have integers $a_1, \dots, a_k$. Then we have for certain integers $m_1, \dots, m_k$ (which are also minors of A_{k+1})

$$\det A_{k+1} = (-1)^{k+2} a_1 m_1 + \ldots - a_k m_k + b \det A_k.$$

As $\det A_k = 1$, if we put $b = 1 - (-1)^{k+2} a_1 m_1 + \ldots + a_k m_k$, we get $\det A_{k+1} = 1$. (Note that [17] Theorem 4.1 implies $b > 0$). □

Example 3.18. Consider the following complexes:



Then $G_1 = G_{\Delta_1} \cong (\mathbb{Z}^2) *_{\mathbb{Z}} (\mathbb{Z}^2)$ and $G_2 = G_{\Delta_2} \cong (\mathbb{Z}^3) *_{\mathbb{Z}^2} (\mathbb{Z}^3)$. For the group G_1 there is no finitely presented G'-fibre product in G_1^2, whereas for G_2 there is some (but not the untwisted one).

4. Metabelian and virtually solvable groups

In this section we are going to consider the problem of whether normal fibre products inherit higher cohomological finiteness properties of the ambient group but we will restrict ourselves to untwisted fibre products and metabelian and virtually solvable groups. For metabelian groups it is very easy to show that things are as expected if one assumes what is known as

the FP_m-conjecture. This conjecture asserts that a metabelian group G is of type FP_m if and only if $\Sigma^m(G)^c$ is m-tame, i.e., if and only if whenever we have $[\chi_1], \ldots, [\chi_m] \in \Sigma^m(G)^c$, $\sum_{i=1}^m \chi_i \neq 0$. It has been proven when $m = 2$ ([10]), when $m = 3$ and G is a split extension of abelian groups ([8]) and for groups of finite Prüfer rank ([1]), see also [21] and [16] for analogous results in the torsion case.

Lemma 4.1. *Let $m, n > 0$ and let G be a metabelian group of type FP_m belonging to a class of groups closed under subgroups and taking direct products where the FP_m-conjecture is known to be true. Then any untwisted normal fibre product H in G^n is also of type FP_m.*

Proof. By Lemma 3.1 we may assume that H is the untwisted G'-fibre product. Let $Q = G/G'$, $A = G'$. By the FP_m-conjecture $\Sigma^1(G)^c = \Sigma_A(Q)^c$ is m-tame. But since $B := H' = A \times \ldots \times A$ and $H/H' \cong Q$, we have $\Sigma^1(H)^c = \Sigma_A(Q)^c$ by [11] Lemma 1.1 c). The result follows again from the FP_m-conjecture. □

In our next result we consider a much more general class of groups but we have to pay the price of restricting the finiteness condition.

Theorem 4.2. *Let G be virtually solvable of type FP_∞. Then for $n > 0$ any untwisted normal fibre product H of G^n is also of type FP_∞.*

Proof. Again, by Lemma 3.1 we only have to consider the untwisted G'-fibre product H. Virtually solvable groups of type FP_∞ are virtually nilpotent-by-abelian (and constructible) ([19], [24]). Let Γ be a finite index normal nilpotent-by-abelian subgroup of G and let H_Γ be the untwisted normal fibre product of Γ, observe that $H_\Gamma \leq H \cap (\Gamma^n)$. Note that $(\Gamma^n)' \leq (G^n)'$ and that Γ is FP_∞. Assume that H_Γ is FP_∞. We claim that also $H \cap (\Gamma^n)$ is. To see it essentially one only has to use [9] Theorem B since for any $\chi \in S(\Gamma^n)$ such that $H \cap (\Gamma^n) \leq \mathrm{Ker}\chi$ we have $H_\Gamma \leq \mathrm{Ker}\chi$ thus $\chi \in \Sigma^\infty(\Gamma^n, \mathbb{Z})$. Now, as $H \cap (\Gamma^n)$ has finite index in H we deduce that H is FP_∞ too. This means that we may assume that G is nilpotent-by-abelian.

From [25] Corollary 5.2 we deduce that

$$\Sigma^\infty(G, \mathbb{Z})^c = \mathrm{conv}\Sigma^1(G)^c \tag{6}$$

where $\mathrm{conv}\Sigma^1(G)^c$ denotes the *convex hull* of $Sigma^1(G)^c$, i.e., $\{[\chi_1 + \ldots + \chi_l] \mid \chi_1, \ldots, \chi_l \in \Sigma^1(G)^c\}$. Moreover, by [1] Proposition IV 2.2 and Theorem IV 1.1, $\Sigma^1(G)^c$ is ∞-tame, i.e. m-tame for any m thus $\Sigma^\infty(G, \mathbb{Z})^c$ is also ∞-tame.

Now, let $(0, \ldots, 0) \neq \chi = (\chi_1, \ldots, \chi_n)$ be a character of G^n and assume that $H \leq \mathrm{Ker}\chi$. This means that for any $g \in G$, $\chi(g, \ldots, g) = \sum \chi_i(g) = 0$. Thus there is some i with $\chi_i \in \Sigma^\infty(G, \mathbb{Z})$.

Meiner's inequality in the product formula ([7] Theorem 1.2) implies that $\chi \in \Sigma^\infty(G, \mathbb{Z})$. Finally, by [9] Theorem B we deduce that H is FP_∞. □

Question 4.3. Let $n, m > 0$ and G virtually solvable of type FP_m. When is the untwisted normal fibre product in G^n also of type FP_m?

Since the key point in the proof of Proposition 4.2 was equation (6), the answer will be yes at least when the analogous formula holds for m. The

subindex $\leq i$ in the statement means that only sums in the convex hull of less than i characters are considered.

Proposition 4.4. *Let $n, m > 0$ and let G be a virtually solvable group of type* FP_m. *Assume that there is some finite index normal subgroup Γ such that for any $0 \leq i < m$,*

$$conv_{\leq i}\Sigma^1(\Gamma)^c = \Sigma^i(\Gamma, \mathbb{Z})^c. \tag{7}$$

Then any untwisted normal fibre product H of G^n is of type FP_m.

Proof. Again, it suffices to consider the case when H is the untwisted normal fibre product of G. With the same argument as at the beginning of the proof of Proposition 4.2, we may assume that $\Gamma = G$, i.e., that the formula of (7) holds true for G. Let $0 \neq \chi = (\chi_1, \dots, \chi_n)$ be a character of G^n with $H \leq \mathrm{Ker}\chi$ thus $\sum \chi_i = 0$. We claim that whenever $0 \leq i_1, \dots, i_n$ are such that $\sum i_j = m$ and $i_j = 0$ if and only if $\chi_j = 0$, there is some l with $[\chi_l] \in \Sigma^{i_l}(G, \mathbb{Z})$ (in particular, $\chi_l \neq 0$). Observe first that we must have $i_j < m$ for $1 \leq j \leq n$ since in other case $\chi_i = 0$ for any $i \neq j$. If the claim were false, we would have for any j with $0 \neq \chi_j$, that $\chi_j \in \Sigma^{i_j}(G, \mathbb{Z})^c = \mathrm{conv}_{\leq i_j}\Sigma^1(G)^c$. Expressing χ_j as a sum of at most i_j characters all in $\Sigma^1(G)^c$ and taking into account that $\sum \chi_i = 0$, we would have a set of at most m characters in $\Sigma^1(G)^c$ summing up 0, a contradiction with the fact that $\Sigma^1(G)^c$ is m-tame ([1] Proposition IV 2.2 and Theorem IV 1.1). The rest of the proof is exactly as in Proposition 4.2

□

Remark 4.5. Let $m > 0$ or $m = \infty$. A group is of type $\mathrm{F_m}$ if and only if it is of type FP_m and finitely presented. So one can write down versions of Proposition 4.2 and 4.4 for $\mathrm{F_m}$ and the same can be done with Question 4.3.

Question 4.6. What can be said for the property of having finitely generated homology groups?

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